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# The Solvability of a Class of Generalized Nonlinear Variational Inequalities Based on an Iterative Algorithm

R. U. VERMA
International Publications
Division of Mathematical Sciences
12046 Coed Drive, Orlando, FL 32826, U.S.A.

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**Abstract**—Based on a new iterative algorithm, the solvability of a class of nonlinear variational inequalities involving a combination of relaxed monotone operators in a Hilbert space setting is presented. 

**Keywords**—Iterative algorithm, Relaxed monotone operators, Nonlinear variational inequalities.

## 1. INTRODUCTION

General variational inequalities represent an important and extremely useful class of nonlinear problems arising from applied mathematics, optimization and control theory, economics, mechanics, engineering sciences, physics, and others. Variational inequalities can be reduced to the complementarity problem by discretization and Lagrange multipliers, which provides an approach to important algorithmic developments.

Our aim in this paper is to apply a general iterative procedure to the solvability of a class of nonlinear variational inequalities involving a combination of relaxed monotone operators in a Hilbert space setting. The obtained result unifies a number of nonlinear variational inequality problems in this direction. For more on nonlinear variational inequalities, we refer to [1–7].

Let $H$ be a real Hilbert space and let $\langle u, v\rangle$ and $\|u\|$ denote, respectively, the inner product and norm on $H$ for all $u, v \in H$. Let $K$ be a nonempty closed convex subset of $H$. Let $A, S, T, g : H \to H$ be nonlinear mappings on $H$ and let $K(\cdot) : H \to P(H)$ be a multivalued mapping. Then the following problem is said to be a nonlinear variational inequality (NVI) problem: determine an element $u$ of $K$ such that:

(i) $g(u) \in K(u)$,

(ii)

$$\langle A(g(u)), v - g(u)\rangle \geq \langle A(u), v - g(u)\rangle - t\left\langle (S+T)(u), v - g(u)\right\rangle, \quad \text{for all } v \in K(u), \tag{1}$$

where $t > 0$ is a constant.

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For $K(u) = K$ for all $u \in H$, the NVI problem [1] reduces to: find an element $u$ of $K$ such that $g(u) \in K$ and

$$\langle A(g(u)), v - g(u)\rangle \geq \langle A(u), v - g(u)\rangle - t\,\langle (S+T)(u), v - g(u)\rangle\,, \qquad \text{for all } v \in K, \tag{2}$$

where $t > 0$ is a constant.

Replacing $g(u)$ by $u_0$ for some $u \in K$ in (2), the NVI problem (2) reduces to [2]: find $u_0$ in $K$ such that

$$\langle Au_0, v - u_0\rangle \geq \langle A(u), v - u_0\rangle - t\langle (S+T)(u), v - u_0\rangle, \qquad \text{for all } v \in K, \tag{3}$$

where $t > 0$ is a constant.

For $S = I$, the identity on $H$, the NVI problem (1) reduces to: find an element $u$ of $K$ such that $g(u) \in K(u)$ and

$$\langle A(g(u)), v - g(u)\rangle \geq \langle A(u), v - g(u)\rangle - t\,\langle (I+T)(u), v - g(u)\rangle\,, \qquad \text{for all } v \in K(u), \tag{4}$$

where $t > 0$ is a constant.

DEFINITION 1.1. *(see [1].) $T : H \to H$ is said to be a relaxed monotone operator if there exists a constant $r > 0$ such that*

$$\langle Tx - Ty, x - y\rangle \geq -r\|x - y\|^2, \qquad \text{for all } x, y \in H. \tag{5}$$

Relaxed operators were applied to the study of constrained problems in reflexive Banach spaces, where the set of all admissible elements is nonconvex but star-shaped. As a result, corresponding variational formulations are no longer variational inequalities but, instead, become hemivariational inequalities [1,3,4].

DEFINITION 1.2. *$T : H \to H$ is called Lipschitz continuous (or Lipschitzian) if there is a constant $s > 0$ such that for all $x, y$ in $H$, we have*

$$\|Tx - Ty\| \leq s\|x - y\|. \tag{6}$$

Clearly, (6) implies

$$\langle Tx - Ty, x - y\rangle \leq s\|x - y\|^2. \tag{7}$$

We note that (7) implies $-T$ is relaxed monotone.

## 2. AUXILIARY AND MAIN RESULTS

In this section, we first consider some auxiliary results, and then present the main result on the solvability of the NVI problem (1).

LEMMA 2.1. *An element $u$ of $K$ is a solution of the NVI problem (1) iff $u \in K$ such that $g(u) \in K(u)$ and*

$$\langle A(g(u)) - A(u) + t(S+T)(u), v - g(u)\rangle \geq 0, \qquad \text{for all } v \in K(u). \tag{8}$$

LEMMA 2.2. *An element $u$ of $K$ is a solution of the NVI problem (1) iff $u \in K$ satisfies $g(u) \in K(u)$ and*

$$g(u) = m(u) + P_K[g(u) + A(u) - A(g(u)) - t(S+T)(u) - m(u)], \tag{9}$$

*where $P_K$ is the projection of $H$ onto $K$.*

ALGORITHM 2.1. For $u_0$ in $H$, $u_{n+1}$ is defined by an iterative scheme

$$u_{n+1} = u_n - g(u_n) + m(u_n) + P_K[g(u_n) + A(u_n) - A(g(u_n)) - t(S+T)(u_n) - m(u_n)] \tag{10}$$

for $n \geq 0$, where $t > 0$ is a constant.

THEOREM 2.1. *Let $H$ be a real Hilbert space, $K$ be a finite closed convex subset of $H$, and the set $K(u) = m(u) + K$ be defined by a multivalued mapping $K(u) : H \to P(H)$, where $m : H \to H$ is Lipschitz continuous with the Lipschitz continuity constant $p > 0$. Let $S : H \to H$ be strongly monotone and Lipschitz continuous with the strong monotonicity constant $d > 0$ and the Lipschitz continuity constant $b > 0$, $T : H \to H$ be Lipschitzian and relaxed monotone with the Lipschitz continuity constant $q > 0$ and the relaxed monotonicity constant $r > 0$, $A : H \to H$ be Lipschitz continuous with the Lipschitz continuity constant $s > 0$, and $g : H \to H$ be strongly monotone and Lipschitz continuous with the strong monotonicity constant $a > 0$ and the Lipschitz continuity constant $c > 0$. Then the NVI problem (1) has a unique solution $u'$, and the approximate solution $u_n \to u'$ (strongly) for all $t > 0$ such that*

$$\left| t - \frac{(d-r)}{(b+q)^2} \right| < \frac{\left[(d-r)^2 - (b+q)^2\left(1-(1-2k)^2\right)\right]^{1/2}}{(b+q)^2},$$

*for $r < d$ and $k < 1/2$, where $k = (1 - 2a + c^2)^{1/2} + p + s(1+c)/2$.*

PROOF. Assume $u' \in K$ is a solution of the NVI problem (1). Then $g(u') \in K(u')$. Since $P_K$ is nonexpansive, by Lemma 2.2, we have

$$g(u') = m(u') + P_K\left[g(u') + A(u') - A(g(u')) - t(S+T)(u') - m(u')\right],$$

and

$$\begin{aligned}
\|u_{n+1} - u'\| &= \|u_n - g(u_n) + m(u_n) + P_K\left[g(u_n) + A(u_n) - A(g(u_n)) - t(S + T(u_n)) - m(u_n)\right] \\
&\quad - (g(u') - g(u') + u')\| \\
&= \|u_n - u' - (g(u_n) - g(u')) + m(u_n) - m(u') \\
&\quad + P_K\left[g(u_n) + A(u_n) - A(g(u_n)) - t(S+T)(u_n) - m(u_n)\right] \\
&\quad - P_k\left[g(u') + A(u') - A(g(u')) - t(S+T)(u') - m(u')\right]\| \\
&\le \|u_n - u' - (g(u_n) - g(u'))\| + \|m(u_n) - m(u')\| \\
&\quad + \|g(u_n) + A(u_n) - A(g(u_n)) - t(S+T)(u_n) - m(u_n) \\
&\quad - \left[g(u') + A(u') - A(g(u')) - t(S+T)(u') - m(u')\right]\| \\
&\le 2\|u_n - u' - (g(u_n) - g(u'))\| + 2\|m(u_n) - m(u')\| + \|A(u_n) - A(u')\| \\
&\quad + \|A(g(u_n)) - A(g(u'))\| + \|u_n - u' - t\left[(S+T)(u_n) - (S+T)(u')\right]\| \\
&\le \left[2\left(1 - 2a + c^2\right)^{1/2} + 2p + s + sc + \left(1 - 2t(d-r) + t^2(b+q)^2\right)^{1/2}\right]\|u_n - u'\| \\
&= (2k + L)\|u_n - u'\| \\
&\le (2k + L)^n\|u_1 - u'\|,
\end{aligned}$$

for $2k + L < 1$ and $k < 1/2$, where $k = (1 - 2a + c^2)^{1/2} + p + (s(1+c)/2)$ and $L = (1 - 2t(d - r) + t^2(b+q)^2)^{1/2}$. Hence $u_n \to u'$ (strongly), and this completes the proof.

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